Chapter 7.1: Integrals and Transcendental Functions

Chapter 7.3: Hyperbolic Functions

2-in-1 special

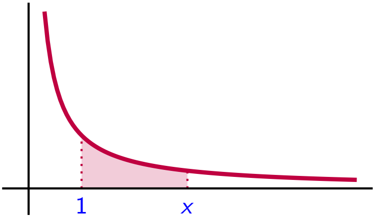
Logarithm as Integral

Old idea — $\ln(x)$ is the inverse of e^x.

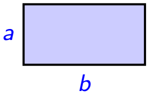
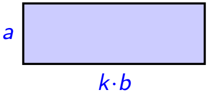
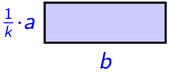
But how do we go about finding e^x?

$$\boxed{\ln(x) = \int_1^x \frac{1}{t}\,dt}$$

e^x is the inverse of $\ln(x)$. We can compute (estimate) $\ln(x)$ by integrals and everything is nicely defined (even for unusual values of x). Some properties of logarithms follow from geometry.



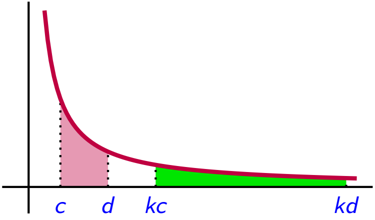
Scaling as a geometric tool

Rectangle	Area
 a b	ab
 a $k \cdot b$	$a \cdot k \cdot b$
 $\frac{1}{k} \cdot a$ b	$\frac{1}{k} \cdot a \cdot b$
 $\frac{1}{k} \cdot a$ $k \cdot b$	ab

$\ln(x) = \int_1^x \frac{1}{t}\, dt$

Scale plane horizontally by k and vertically by $\frac{1}{k}$. Then the area (and so integral) is unchanged.

In this scaling, curve $y = \frac{1}{x}$ goes to the curve $y = \frac{1}{x}$. So the following two shaded area are equal.



Example: Using properties of integration, show that $\ln(1) = 0$.

$$\ln(x) = \int_1^x \frac{1}{t}\, dt = \int_1^1 \frac{1}{t}\, dt = 0$$

Example: Use properties of integration and scaling of areas to show $\ln(\frac{1}{c}) = -\ln(c)$ and $\ln(ab) = \ln(a) + \ln(b)$.

$$\ln\left(\frac{1}{c}\right) = \int_1^{\frac{1}{c}} t\, dt = \int_c^1 t\, dt = -\int_1^c t\, dt = -\ln\left(c\right)$$

The second $=$ is scaling the plane by c.

$$\ln(a) + \ln(b) = \int_1^a t\, dt + \int_1^b t\, dt = \int_1^a t\, dt + \int_a^{ab} t\, dt = \int_1^{ab} t\, dt = \ln\left(ab\right)$$

The second $=$ is scaling the plane by a for the second integral.

$\ln(x) = \int_1^x \frac{1}{t}\,dt$

By the fundamental theorem of calculus

$$\frac{d}{dx}\left(\ln(x)\right) = \frac{d}{dx}\left(\int_1^x \frac{1}{t}\,dt\right) = \frac{1}{x}.$$

As a consequence we have the following general rule for antiderivatives

$$\boxed{\int \frac{f'(x)}{f(x)}\,dx = \ln\left(f(x)\right) + C}$$

Example: Find $\int \tan\theta\,d\theta$.

$$\int \tan\theta\,d\theta = \int \frac{\sin\theta}{\cos\theta}\,d\theta = \int -\frac{1}{u}\,du$$

Substitution $u = \cos\theta,\ du = -\sin\theta d\theta$

$$\int -\frac{1}{u}\,du = -\ln|u| + C = -\ln|\cos\theta| + C = \ln|\cos(\theta)^{-1}| + C = \ln|\sec(\theta)| + C$$

Example: Find $\int \sec\theta\,d\theta$.

(Hint: for what functions does $\sec\theta$ appear in the derivative?)

$\frac{d}{d\theta}(\tan\theta) = \sec^2\theta = \sec(\theta)\sec(\theta)$

$\frac{d}{d\theta}(\sec\theta) = \sec(\theta)\tan(\theta)$

$\frac{d}{d\theta}(\tan\theta + \sec\theta) = \sec(\theta)(\tan(\theta) + \sec(\theta))$

$$\int \sec\theta\,d\theta = \int \sec\theta\cdot\frac{\tan(\theta)+\sec(\theta)}{\tan(\theta)+\sec(\theta)}\,d\theta$$

This has form $\int \frac{f'(x)}{f(x)}\,dx = \ln\left(f(x)\right) + C$

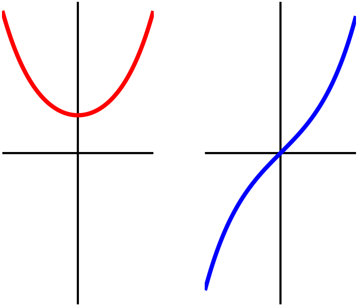
Hence

$$\int \sec\theta\,d\theta = \ln\left(\tan(\theta) + \sec(\theta)\right) + C$$

Hyperbolic functions

$$\cosh(x) = \frac{e^x + e^{-x}}{2} \qquad \leftarrow \text{ even part of } e^x$$

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \qquad \leftarrow \text{ odd part of } e^x$$



The hyperbolic cosine ($\cosh(x)$) curve is an important curve with real-world applications. In particular, if you look at the shape of a hanging chain it forms a *catenary* which is the hyperbolic cosine curve.

Example: Find

$$\frac{d}{dx}\big(\sinh(x)\big) =$$

$$\frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right) = \frac{e^x + e^{-x}}{2} = \cosh(x)$$

$$\frac{d}{dx}\big(\cosh(x)\big) =$$

$$\frac{d}{dx}\left(\frac{e^x + e^{-x}}{2}\right) = \frac{e^x - e^{-x}}{2} = \sinh(x)$$

Notice $\frac{d^{100}}{dx^{100}}\big(\cosh(x)\big) = \cosh(x)$

Hyperbolic functions

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} \qquad \operatorname{sech}(x) = \frac{1}{\cosh(x)}$$

- $\cosh^2(x) + \sinh^2(x) =$ $\left(\frac{e^x+e^{-x}}{2}\right)^2 + \left(\frac{e^x-e^{-x}}{2}\right)^2 = \frac{e^{2x}+2+e^{-2x}}{4} + \frac{e^{2x}-2+e^{-2x}}{4} = \cosh(2x)$
- $\cosh^2(x) - \sinh^2(x) =$ $\left(\frac{e^x+e^{-x}}{2}\right)^2 - \left(\frac{e^x-e^{-x}}{2}\right)^2 = \frac{e^{2x}+2+e^{-2x}}{4} - \frac{e^{2x}-2+e^{-2x}}{4} = \frac{4}{4} = 1$
- $2\sinh(x)\cosh(x) = 2 \cdot \frac{e^x-e^{-x}}{2} \cdot \frac{e^x+e^{-x}}{2} = \frac{e^{2x}-e^{-2x}}{2} = \sinh(2x)$
- $\frac{d}{dx}(\tanh(x)) = \frac{d}{dx}\left(\frac{\sinh(x)}{\cosh(x)}\right) = \frac{\cosh(x)^2 - \sinh(x)^2}{\cosh(x)^2} = \frac{1}{\cosh(x)^2}$
- $\frac{d}{dx}(\operatorname{sech}(x)) = \frac{d}{dx}\left(\frac{1}{\cosh(x)}\right) = -\frac{1}{\cosh(x)^2} \cdot \sinh(x) = -\operatorname{sech}(x)\tanh(x)$

(Note these bear a resemblance to trig functions. Not a coincidence!!)